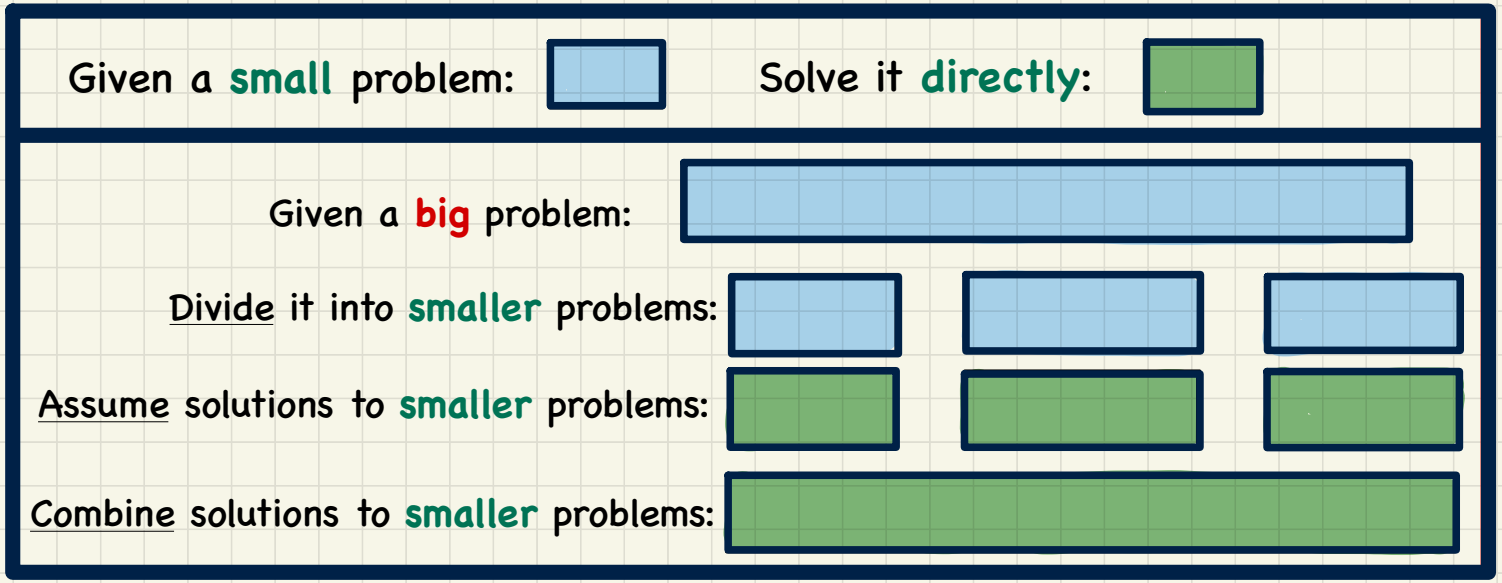
Lecture 8

Part A

Recursion -
Basics: Thinking Recursively, Call Stack

Solving a Problem Recursively




```
m(i) {
  if(i == ...) { /* base case: do something directly */ }
  else {
    m(j);/* recursive call with strictly smaller value */
  }
}
```

recursive call of m

m(100)
↳ m(j)
j < 100

Recursive Solution: factorial

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 & \text{if } n \geq 1 \end{cases}$$

$5! = 5 * 4 * 3 * 2 * 1$

↳ $4!$

$= 5 * 4!$

$= 5 * (5-1)!$

↳ strictly smaller problem size

$n-1 < n$.

$n! = n * (n-1) * (n-2) * \cdots * 2 * 1$

↳ $(n-1)!$

$= n * (n-1)!$

Recursive Solution in Java: factorial

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

strictly smaller problem

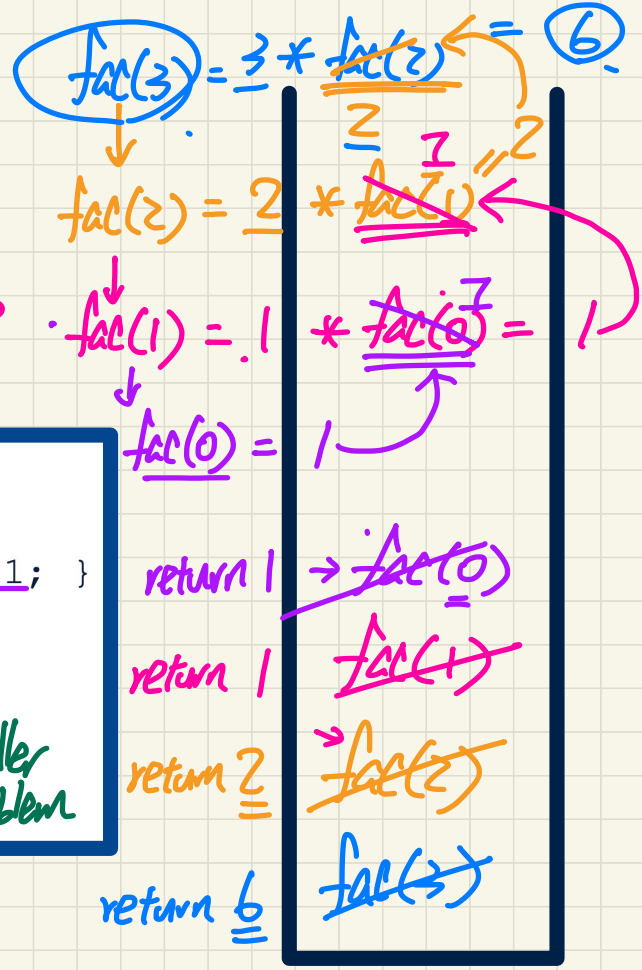
base case

recursive case

```
int factorial (int n) {
  int result;
  if (n == 0) { /* base case */ result = 1; }
  else { /* recursive case */
    result = n * factorial (n - 1);
  }
  return result;
}
```

recursive call

strictly smaller problem



Example: factorial(3)

Common Errors of Recursion (1)

```
int factorial (int n) {
  return n * factorial (n - 1);
}
```

↳ missing base case(s)

Infinite Recursion

fac(3)
↳ fac(2)
↳ fac(1)
↳ fac(0)
↳ fac(-1)
⋮

Common Errors of Recursion (2)

```
int factorial (int n) {
  if(n == 0) { /* base case */ return 1; }
  else { /* recursive case */ return n * factorial (n); }
}
```

Infinite Recursion

fac(3)
↳ fac(3)
↳ fac(3)
↳ fac(3)
⋮

(never able to reach the base case)

problem size for recursive call is not strictly smaller.

Recursive Solution: Fibonacci Numbers

F_7 F_8 F_9

F = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Base Cases

$F_1 = 1$

$F_2 = 1$

Recursive Cases

$F_n = F_{n-1} + F_{n-2}$

$n > 2$

$n-1$ strictly smaller than n

$n-2 < n$

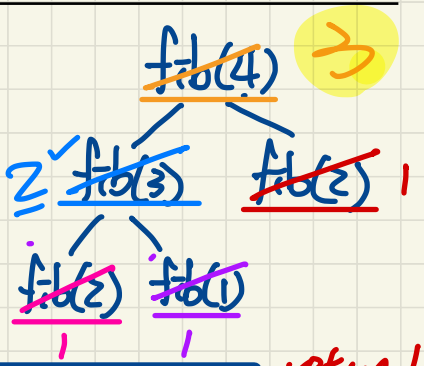
solved recursively by two recursive calls

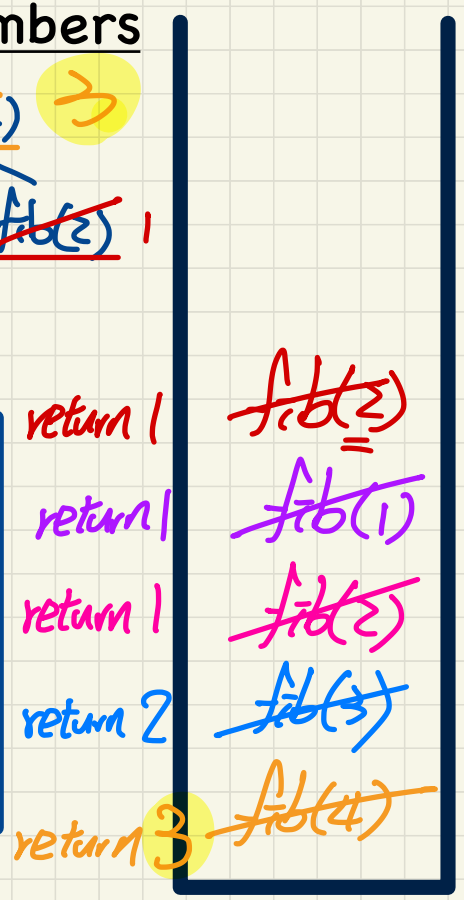
$F_9 = F_7 + F_8$.

Recursive Solution in Java: Fibonacci Numbers

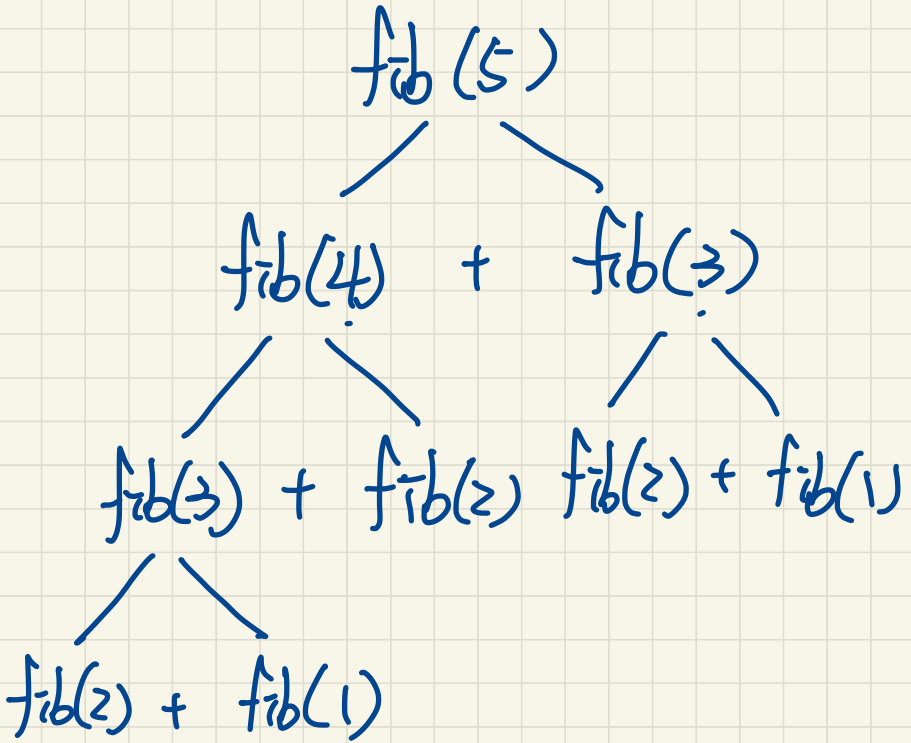
$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-1} + F_{n-2} & \text{if } n > 2 \end{cases}$$

```
int fib (int n) {
  int result;
  if(n == 1) { /* base case */ result = 1; }
  else if(n == 2) { /* base case */ result = 1; }
  else { /* recursive case */
    result = fib (n - 1) + fib (n - 2);
  }
  return result;
}
```





Example: fib(4)


fib(5)
fib(4) + fib(3)
fib(3) + fib(2) fib(2) + fib(1)
fib(2) + fib(1)

Lecture 7

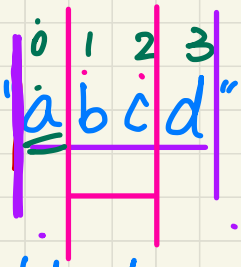
Part B

Recursion -
Examples: Recursions on Strings

Use of String

```
public class StringTester {
  public static void main(String[] args) {
    String s = "abcd";
    System.out.println(s.isEmpty()); /* false */
    /* Characters in index range [0, 0) */
    String t0 = s.substring(0, 0);
    System.out.println(t0); /* "" */
    /* Characters in index range [0, 4) */
    String t1 = s.substring(0, 4);
    System.out.println(t1); /* "abcd" */
    /* Characters in index range [1, 3) */
    String t2 = s.substring(1, 3);
    System.out.println(t2); /* "bc" */
    String t3 = s.substring(0, 2) + s.substring(2, 4);
    System.out.println(s.equals(t3)); /* true */
    for(int i = 0; i < s.length(); i ++) {
      System.out.print(s.charAt(i));
    }
    System.out.println();
  }
}
```

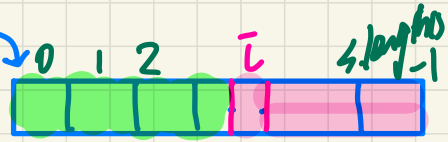
substring(n, m) [n, m) = [n, m-1]



Assume i valid index

s.substring(0, i) + s.substring(i, s.length())

→ s

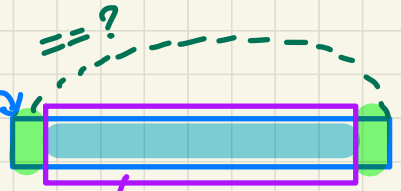


s.charAt(0) → 'a'

s.charAt(s.length()-1) → 'd'

Recursions on Strings

Palindrome

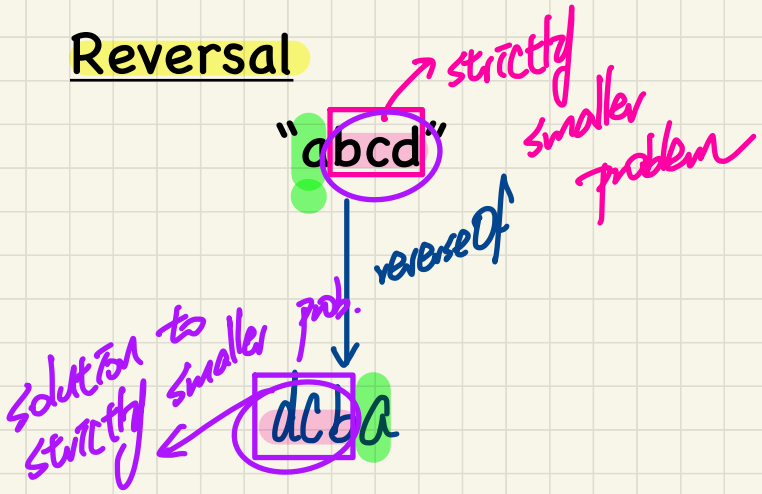


"racecar" → T

"aracecars" → F

"raceacar" → F

Reversal



Number of Occurrences

"abca"

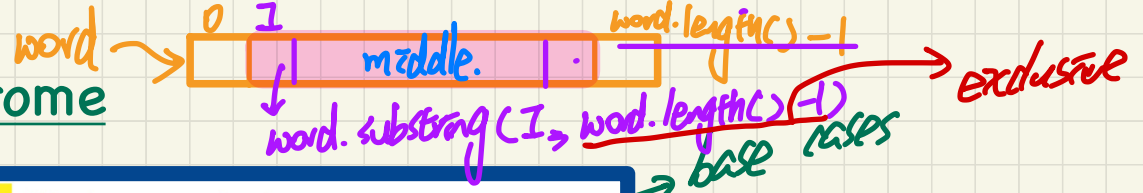
'a' 2 = 1 + 1

'b' 1 = 0 + 1

# of occurrences of the char in tail of input string.

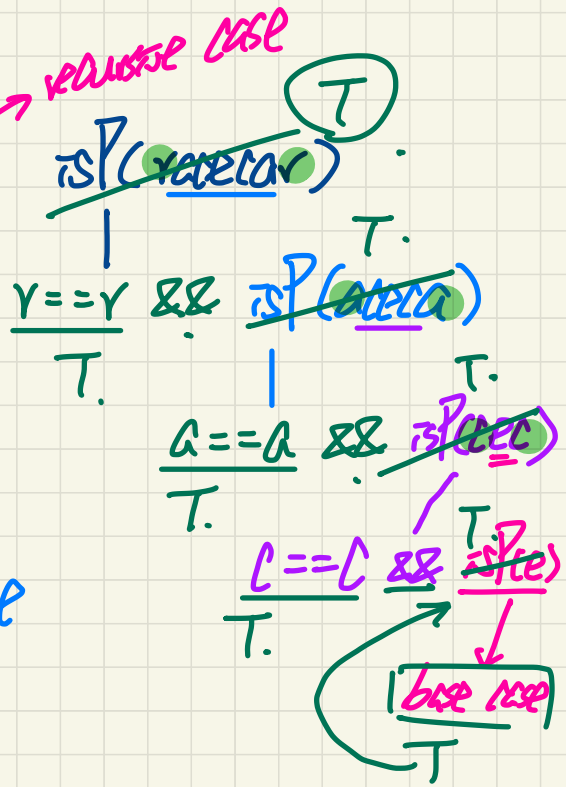
is the char equal to the head of string

Problem: Palindrome




```
boolean isPalindrome (String word) {
  if(word.length() == 0 || word.length() == 1) {
    /* base case */
    return true;
  }
  else {
    /* recursive case */
    char firstChar = word.charAt(0);
    char lastChar = word.charAt(word.length() - 1);
    String middle = word.substring(1, word.length() - 1);
    return
        firstChar == lastChar
        /* See the API of java.lang.String.substring. */
        && isPalindrome (middle);
  }
}
```

recursive call to solve a subproblem with strictly smaller size

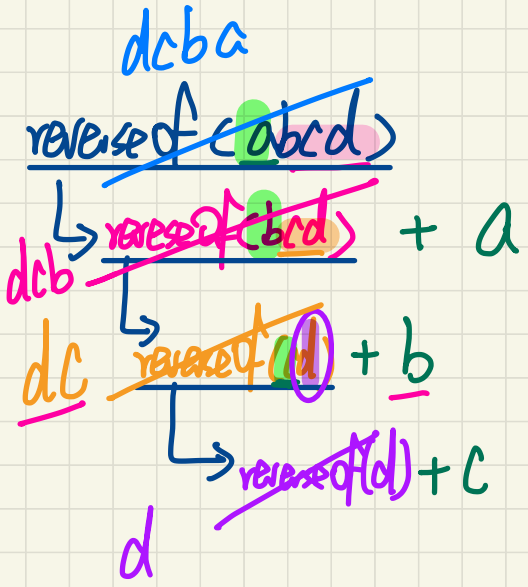


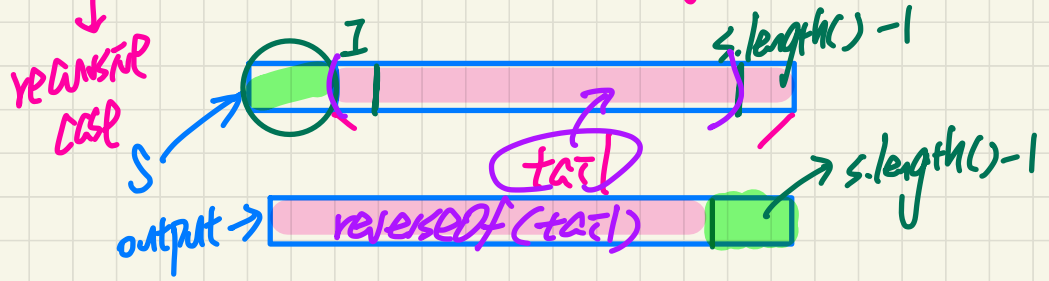
Problem: Reverse of a String

```
String reverseOf (String s) {
  if(s.isEmpty()) { /* base case 1 */
    return "";
  }
  else if(s.length() == 1) { /* base case 2 */
    return s;
  }
  else { /* recursive case */
    String tail = s.substring(1, s.length());
    String reverseOfTail = reverseOf (tail);
    char head = s.charAt(0);
    return reverseOfTail + head;
  }
}
```

base cases

recursive call to solve a strictly smaller problem.





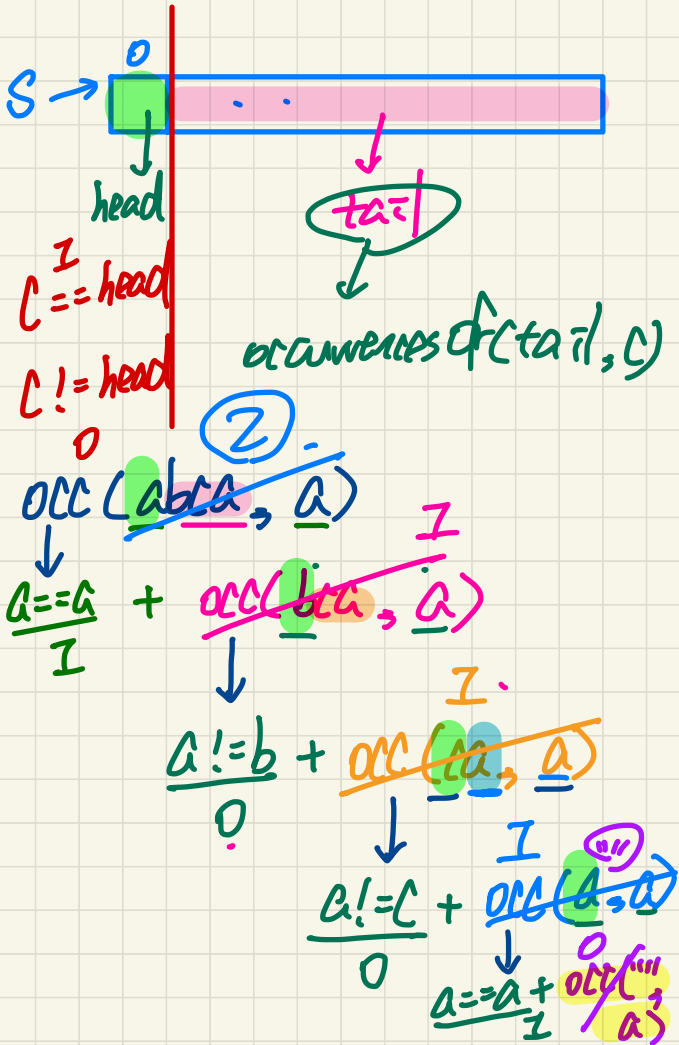
Problem: Number of Occurrences

```
int occurrencesOf(String s, char c) {
  if(s.isEmpty()) {
    /* Base Case */
    return 0;
  }
  else {
    /* Recursive Case */
    char head = s.charAt(0);
    String tail = s.substring(1, s.length());
    if(head == c) {
      return 1 + occurrencesOf(tail, c);
    }
    else {
      return 0 + occurrencesOf(tail, c);
    }
  }
}
```

base case

recursive case

what if s is "a"? → "" (empty)



Lecture 7

Part C

Recursion -
Examples: Recursions on Arrays

Recursion on an Array: Passing new Sub-Arrays

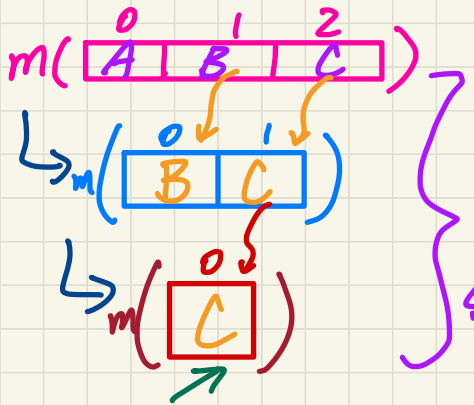
```
void m(int[] a) {
  if(a.length == 0) { /* base case */ }
  else if(a.length == 1) { /* base case */ }
  else {
    int[] sub = new int[a.length - 1];
    for(int i = 1; i < a.length; i ++) { sub[i-1] = a[i]; }
    m(sub); } }
```

base cases

recursive case

Say a1 = {} consider m(a1) → execute the base case

Say a2 = {A, B, C}, consider m(a2)



Recursion on an Array: Passing Same Array Reference

```
void m(int[] a, int from, int to) {
  if (from > to) { /* base case */ }
  else if (from == to) { /* base case */ }
  else { m(a, from + 1, to) } }
```

array of length 1.

base cases

recursive case

empty array.

[0, -1] → empty range.

Say a1 = {}, consider m(a1, 0, a1.length - 1)

m(a1, 0, -1)

from to

min index

max index

Say a2 = {A, B, C}, consider m(a2, 0, a2.length - 1)

m(a2, 0, 2)

m(a2, 1, 2)

strictly smaller problem (elements from indices 1 to 2)

strictly smaller problem (last item in array).

m(2, 2)

Problem: Are All Numbers Positive?

```
boolean allPositive(int[] a) {
  return allPositiveHelper(a, 0, a.length - 1);
}

boolean allPositiveHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return a[from] > 0;
  }
  else { /* recursive case */
    return a[from] > 0 && allPositiveHelper(a, from + 1, to);
  }
}
```

min index

max index

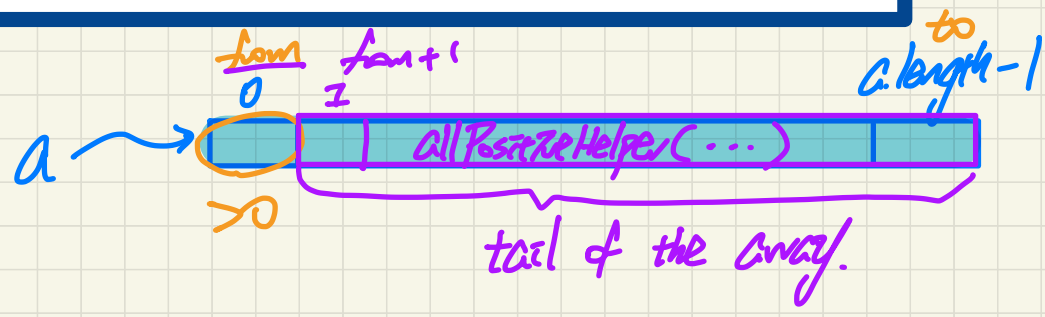
recursive helper method

empty array

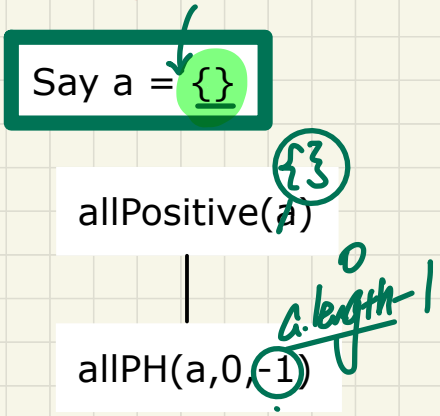
array of length 1

base cases

recursive case



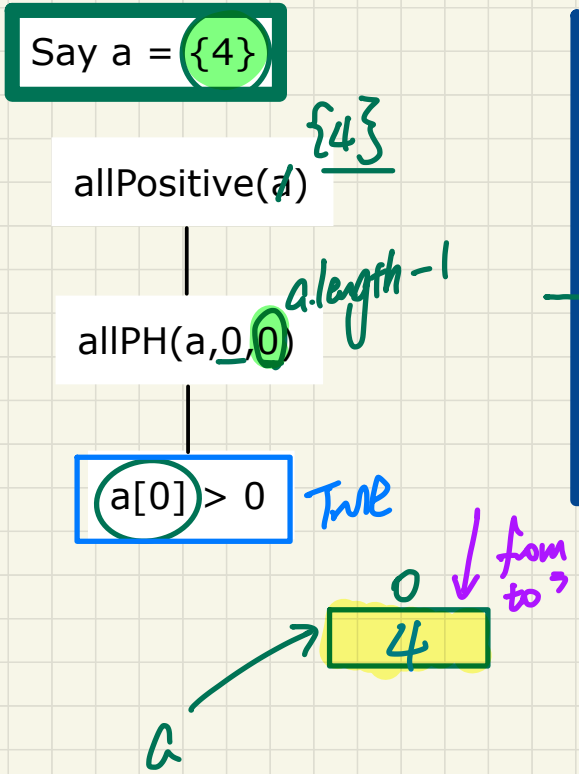
Tracing Recursion: allPositive




```
boolean allPositive(int[] a) {
  return allPositiveHelper(a, 0, a.length - 1);
}

boolean allPositiveHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return a[from] > 0;
  }
  else { /* recursive case */
    return a[from] > 0 && allPositiveHelper(a, from + 1, to);
  }
}
```

Tracing Recursion: allPositive

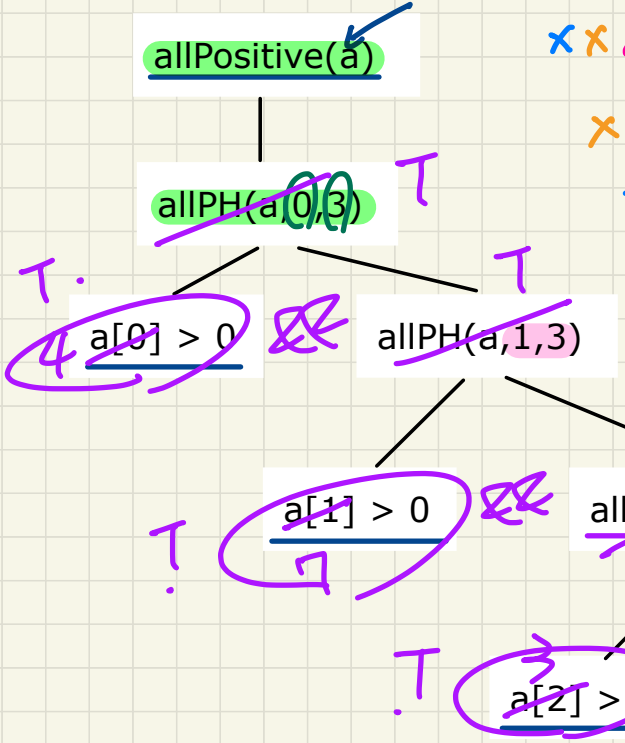



```
boolean allPositive(int[] a) {
  return allPositiveHelper(a, 0, a.length - 1);
}

boolean allPositiveHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return a[from] > 0;
  }
  else { /* recursive case */
    return a[from] > 0 && allPositiveHelper(a, from + 1, to);
  }
}
```

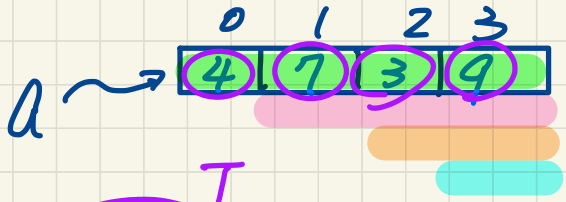
Tracing Recursion: allPositive

Say a = {4,7,3,9}



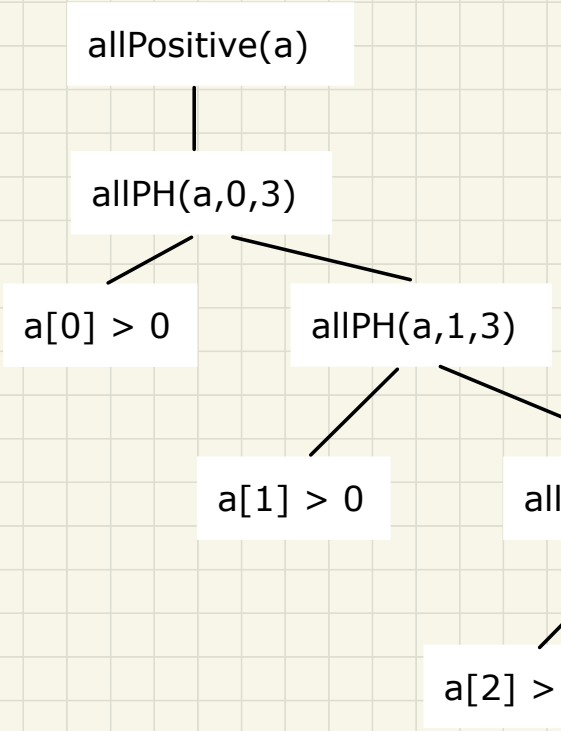

```
boolean allPositive(int[] a) {
  return allPositiveHelper(a, 0, a.length - 1);
}

boolean allPositiveHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return a[from] > 0;
  }
  else { /* recursive case */
    return a[from] > 0 && allPositiveHelper(a, from + 1, to);
  }
}
```



Tracing Recursion: allPositive

Say a = {5,3,-2,9}




```
boolean allPositive(int[] a) {
  return allPositiveHelper(a, 0, a.length - 1);
}

boolean allPositiveHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return a[from] > 0;
  }
  else { /* recursive case */
    return a[from] > 0 && allPositiveHelper(a, from + 1, to);
  }
}
```

Exercise: Trace!

Problem: Are Numbers Sorted?

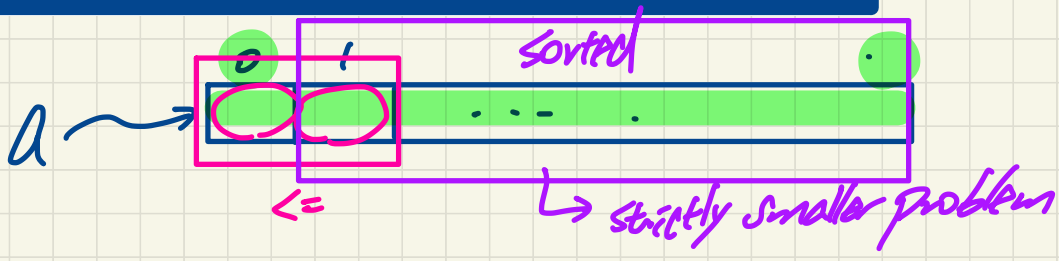
```
boolean isSorted(int[] a) {
  return isSortedHelper(a, 0, a.length - 1);
}

boolean isSortedHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return true;
  }
  else {
    return a[from] <= a[from + 1]
      && isSortedHelper(a, from + 1, to);
  }
}
```

recursive helper method.

base cases

recursive case



Tracing Recursion: isSorted

Say a = {}

isSorted(a)

isSH(a,0,-1)

```
boolean isSorted(int[] a) {
  return isSortedHelper(a, 0, a.length - 1);
}

boolean isSortedHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return true;
  }
  else {
    return a[from] <= a[from + 1]
      && isSortedHelper(a, from + 1, to);
  }
}
```

Tracing Recursion: isSorted

Say a = {4}

isSorted(a)

isSH(a,0,0)

return true

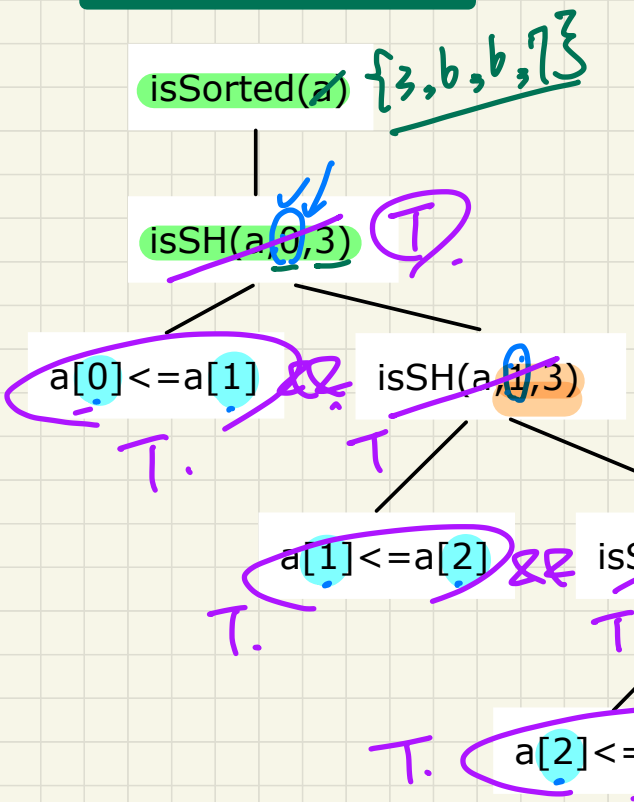
{4}

```
boolean isSorted(int[] a) {
  return isSortedHelper(a, 0, a.length - 1);
}

boolean isSortedHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return true;
  }
  else {
    return a[from] <= a[from + 1]
      && isSortedHelper(a, from + 1, to);
  }
}
```

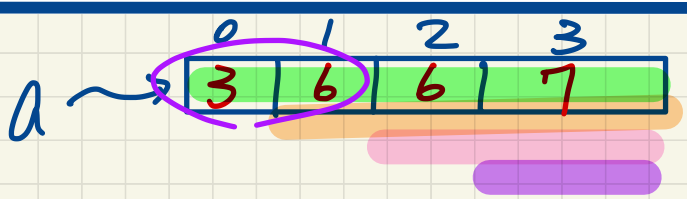
Tracing Recursion: isSorted

Say a = {3,6,6,7}



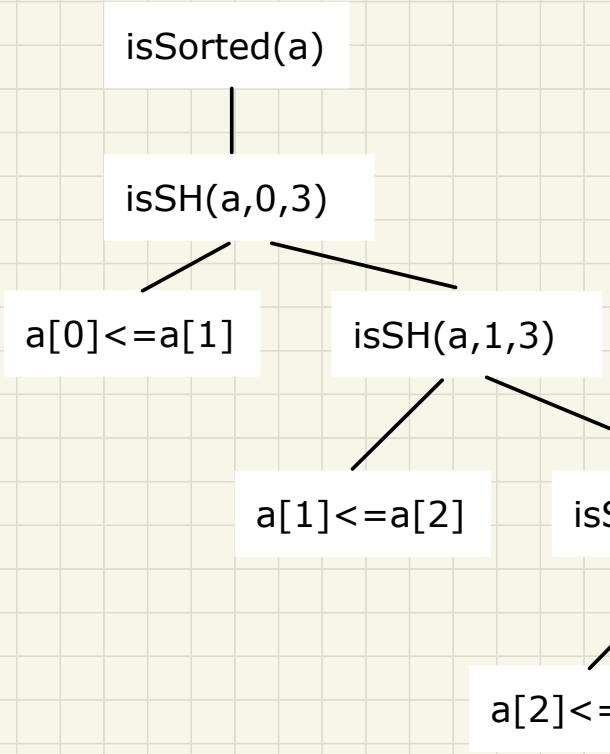

```
boolean isSorted(int[] a) {
  return isSortedHelper(a, 0, a.length - 1);
}

boolean isSortedHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if from == to { /* base case 2: range of one element */
    return true;
  }
  else {
    return a[from] <= a[from + 1]
    && isSortedHelper(a, from + 1, to);
  }
}
```



Tracing Recursion: isSorted

Say a = {3,6,5,7} → F.




```
boolean isSorted(int[] a) {
  return isSortedHelper(a, 0, a.length - 1);
}

boolean isSortedHelper(int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return true;
  }
  else {
    return a[from] <= a[from + 1]
      && isSortedHelper(a, from + 1, to);
  }
}
```

Exercise: Trace